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Eigenfunction Expansions for Symmetric Systems of First Order in the Half-Space R^n_+

by

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1. Introduction

Eigenfunction expansion theory by distorted plane waves was initiated by Ikebe [1] and has been investigated by many authors, for example, Shizuta [6], Shenk II [5], Mochizuki [3], Schulenberger and Wilcox [4] and others. We are concerned with stationary problems for symmetric hyperbolic systems with constant coefficients in the half-space R^n_+ and give an expansion theorem by improper eigenfunctions for such a problem. We note that this problem cannot be treated as a perturbation of whole space problem. In fact, our result is a generalization of the sine and cosine transformations in the L^2 space on the positive half-line which are eigenfunction expansions for $-d^2/dx^2$ in $(0,\infty)$ with Dirichlet or Neumann conditions at $x=0$.

Let R^n denote the n-dimensional Euclidean space. Denote by x the generic point of R^n and write $x'=(x_1,\cdots,x_{n-1})$. We shall also denote by R^n_+ the half-space $\{x=(x',x_n)\varepsilon R^n;\ x_n>0\}$ and by t the time variable. Let L be a first order symmetric hyperbolic operator with constant coefficients:

$$L = I\partial/\partial t - \Sigma_{j=1}^{n} A_j\partial/\partial x_j, \tag{1}$$

where I is the identity matrix of order N and the A_j are $N\times N$ constant Hermitian matrices. We consider the mixed initial and boundary value problem in R_+^n for the operator L:

$$\begin{cases} Lu(t,x) = f(t,x), \quad t>0, \quad x\varepsilon R_+^n, \\ u(0,x) = u_0(x), \quad x\varepsilon R_+^n, \\ Bu(t,x)|_{x_n=0} = 0, \quad t>0, \end{cases} \tag{2}$$

where $u(t,x)$, $f(t,x)$ and $u_0(x)$ are vector-valued functions whose values lie in the N-dimensional complex space C^N and B is an $\ell\times N$ constant matrix with rank ℓ. Replacing $u(t,x)$ and $f(t,x)$ in (2) by $e^{ikt}v(x)$ and $-ie^{ikt}g(x)$, respectively, we obtain the corresponding stationary problem:

$$\begin{cases} (A - kI)v(x) = g(x), \quad x\varepsilon R_+^n, \\ Bv(x)|_{x_n=0} = 0, \end{cases} \tag{3}$$

where

$$A = i^{-1}\Sigma_{j=1}^{n} A_j\partial/\partial x_j. \tag{4}$$

Our aim is to expand an arbitrary function in $L^2(R_+^n)$ by means of generalized or improper eigenfunctions for the self-adjoint operator associated with this problem under some suitable conditions for L(or A) and B.

Let $p(\lambda,\eta)$ be the characteristic polynomial associated with the operator L:

(5) $$p(\lambda,\eta) = \det\ (\lambda I - A(\eta)),$$

where η denotes a generic point of the real dual space Ξ^n of R^n by the duality $x\cdot\eta=x_1\eta_1+\cdots+x_n\eta_n$ and

(6) $$A(\eta) = \Sigma_{j=1}^{n}\ \eta_j A_j.$$

The polynomials $p(\lambda,\eta)$ has a factorization

(7) $$p(\lambda,\eta) = Q_1(\lambda,\eta)^{m_1}\cdots Q_q(\lambda,\eta)^{m_q},$$

where the factors $Q_j(\lambda,\eta)$ are distinct homogeneous polynomials in (λ,η), irreducible over the complex number field C. Since the coefficient of λ^N in $p(\lambda,\eta)$ is 1, the factors are unique, apart from their order, by requiring the coefficient of the highest power of λ in each $Q_j(\lambda,\eta)$ be 1. Put

(8) $$Q(\lambda,\eta) = Q_1(\lambda,\eta)\cdots Q_q(\lambda,\eta).$$

<u>Definition 1</u>. The operator L is called uniformly propagative if the roots $\lambda_j(\eta)$, $1\le j\le\mu$, of $Q(\lambda,\eta)=0$ satisfy the following con-

ditions where μ is the order of $Q(\lambda,\eta)$: (i) The roots $\lambda_j(\eta)$ are all distinct for every η with $|\eta|=1$. (ii) A root function $\lambda_j(\eta)$ vanishes for some real $\eta\neq 0$ if and only if it vanishes identically (see [7]).

Now we state precisely the assumptions that we impose on L and B:

(L.1) The operator L is uniformly propagative.

(L.2) The operator A is elliptic, i.e. $p(0,\eta)\neq 0$ for any η in Ξ^n with $|\eta|=1$.

(L.3) For any real $\lambda\neq 0$ and any $\xi\varepsilon\Xi^{n-1}$ the real roots of $Q(\lambda,\xi,\tau)=0$ with respect to τ are at most double and the number of the real double roots for arbitrarily fixed $(\lambda,\xi)\neq(0,0)$ is at most one.

(B.1) The boundary matrix B is minimally conservative, i.e. $A_n\zeta\cdot\bar{\zeta}=0$ for any $\zeta\varepsilon\mathcal{B}=\ker B\subset C^N$ and if $\mathcal{E}$ is a subspace of C^N such that $\mathcal{E}\supset\mathcal{B}$ and $A_n\zeta\cdot\bar{\zeta}=0$ for any $\zeta\varepsilon\mathcal{E}$, $\mathcal{B}=\mathcal{E}$ holds.

<u>Remark 2</u>. The conditions (L.1) and (L.2) imply that the distinct characteristic roots $\lambda_j(\eta)$, $1\leq j\leq\mu$, of the matrix $A(\eta)$ have constant multiplicities and that μ is even. Thus we put $\mu=2\rho$ and can label $\{\lambda_j(\eta)\}$ in decreasing order:

$$(9)\qquad \begin{cases} \lambda_1(\eta) > \lambda_2(\eta) > \cdots > \lambda_\rho(\eta) > 0 > \lambda_{\rho+1}(\eta) > \cdots > \lambda_{2\rho}(\eta), \\ \lambda_{j+\rho}(\eta) = -\lambda_{\rho-j+1}(-\eta), \quad 1\le j\le\rho, \ \eta\ne0. \end{cases}$$

Moreover we see that N is even. Thus we put $N=2m$. The condition (B.1) implies that $\ell=m$.

Remark 3. The differential operator A defines an unbounded linear operator $\mathcal{A}$ in $L^2(R^n_+)$ with domain

$$D(\mathcal{A}) = \{v(x)\varepsilon C_0^\infty(\overline{R^n_+});\ Bv(x)|_{x_n=0} = 0\}.$$

$\mathcal{A}$ is closable and we denote by $\mathbb{A}$ its closure. Then the condition (B.1) implies that $\mathbb{A}$ is a self-adjoint operator in $L^2(R^n_+)$.

2. Eigenfunctions

Let $G(x,y;\lambda)$ be the Green function for $(\mathbb{A} - \lambda)$, Im $\lambda\ne0$, constructed in [2]. We define projections $P_j(\eta)$, $1\le j\le 2\rho$, by

$$(10)\qquad P_j(\eta) = \begin{cases} \frac{1}{2\pi i}\int_{|\lambda-\lambda_j(\eta)|=\delta} (\lambda I-A(\eta))^{-1}d\lambda, & \eta\ne0, \\ 0, & \eta=0, \end{cases}$$

where δ is chosen sufficiently small such that the set $\{\lambda;\ |\lambda-\lambda_j(\eta)|<\delta\}$ contains no roots of $Q(\lambda,\eta)=0$ except $\lambda_j(\eta)$.

Definition 4. Let $x\varepsilon R^n_+$, $\eta\varepsilon\Xi^n$ and Im $\lambda\ne0$. Define

$$(11)\qquad \Psi_j(x,\eta;\lambda) = \overline{\mathcal{F}}_y[G(x,y;\lambda)](\eta)(\lambda_j(\eta)-\lambda)P_j(\eta),$$

$$(12)\qquad \Psi_j^{\pm}(x,\eta) = \Psi_j(x,\eta;\lambda_j(\eta)\pm i0), \quad 1\le j\le 2\rho,$$

and

$$(13)\qquad \Psi_{j+2\nu\rho}(x,\eta;\lambda) = \frac{\lambda-k_\nu(\xi)}{\lambda-\lambda_j(\eta)}\,\Psi_j(x,\eta;\lambda) \quad \text{for } \eta\varepsilon D_\nu\times\Xi,$$

$$(14)\qquad \Psi_{j+2\nu\rho}^{\pm}(x,\eta) = \Psi_{j+2\nu\rho}(x,\eta;k_\nu(\xi)\pm i0), \quad 1\le j\le 2\rho,$$

$$1\le\nu\le s \text{ for almost every } \eta\varepsilon D_\nu\times\Xi,$$

where the set $\{k_\nu(\xi)\}_{\nu\varepsilon\{j;\ \xi\varepsilon D_j\}}$ is the totality of non-vanishing zeros of the Lopatinski determinant for the system $\{A,B\}$ and $k_i(\xi)\neq k_j(\xi)$ for $\xi\varepsilon D_i\cap D_j$ and $i\neq j$ (see [8]). Here we define $G(x,y;\lambda)=0$ for $x\varepsilon R_+^n$ and $y\notin R_+^n$.

$\Psi_j^{\pm}(x,\eta)$, $\Psi_{j+2\nu\rho}^{\pm}(x,\eta)$ are (improper) eigenfunctions for the operator $\mathbb{A}$, i.e.

$$(15)\qquad \begin{cases} A_x\Psi_j^{\pm}(x,\eta) = \lambda_j(\eta)\Psi_j^{\pm}(x,\eta), \\ B\Psi_j^{\pm}(x,\eta)|_{x_n=0} = 0, \quad 1\le j\le 2\rho, \end{cases}$$

$$(16)\qquad \begin{cases} A_x\Psi_{j+2\nu\rho}^{\pm}(x,\eta) = k_\nu(\xi)\Psi_{j+2\nu\rho}^{\pm}(x,\eta), \quad 1\le j\le 2\rho,\ 1\le\nu\le s, \\ B\Psi_{j+2\nu\rho}^{\pm}(x,\eta)|_{x_n=0} = 0, \qquad \text{for almost every } \eta\varepsilon D_\nu\times\Xi. \end{cases}$$

3. Expansion theorem

Theorem 5. Assume that the conditions (L.1) - (L.3) and (B.1) are satisfied and that $f\varepsilon L^2(R_+^n)$.

(i) The expansion formula

$$(17) \qquad Pf(x) = \Sigma_{j=1}^{2\rho} \int_{\Xi^n} \Psi_j^{\pm}(x,\eta)\hat{f}_j^{\pm}(\eta)d\eta + \Sigma_{\nu=1}^{s} \Sigma_{j=1}^{2\rho} \int_{D_\nu\times\Xi} \Psi_{j+2\nu\rho}^{\pm}(x,\eta)\hat{f}_{j+2\nu\rho}^{\pm}(\eta)d\eta$$

holds, where

$$(18) \qquad \hat{f}_j^{\pm}(\eta) = \int_{R_+^n} \Psi_j^{\pm}(x,\eta)^* f(x)dx, \quad 1\le j\le 2\rho,$$

$$(19) \qquad \hat{f}_{j+2\nu\rho}^{\pm}(\eta) = \int_{R_+^n} \Psi_{j+2\nu\rho}^{\pm}(x,\eta)^* f(x)dx, \quad 1\le j\le 2\rho,\ 1\le\nu\le s.$$

Here the above integrals are taken in the sense of limit in the mean and P is the orthogonal projection onto $R(A)^a = N(A)^{\perp}$.

(ii) $f\varepsilon D(A)$ if and only if $\lambda_j(\eta)\hat{f}_j^{\pm}(\eta)\varepsilon P_j(\eta)L^2(\Xi^n)$, $k_\nu(\xi)\hat{f}_{j+2\nu\rho}^{\pm}(\eta)$ $\varepsilon P_j(\eta)L^2(D_\nu\times\Xi)$, $1\le j\le 2\rho$, $1\le\nu\le s$. Then

$$(20) \qquad (\mathbb{A}f)(x) = \Sigma_{j=1}^{2\rho} \int_{\Xi^n} \lambda_j(\eta)\Psi_j^{\pm}(x,\eta)\hat{f}_j^{\pm}(\eta)d\eta + \Sigma_{\nu=1}^{s} \Sigma_{j=1}^{2\rho} \int_{D_\nu\times\Xi} k_\nu(\xi)\Psi_{j+2\nu\rho}^{\pm}(x,\eta)\hat{f}_{j+2\nu\rho}^{\pm}(\eta)d\eta,$$

$$(21) \qquad (\mathbb{A}f)_j^{\wedge\pm}(\eta) = \lambda_j(\eta)\hat{f}_j^{\pm}(\eta), \quad 1\le j\le 2\rho,$$

$$(22) \qquad (\mathbb{A}f)_{j+2\nu\rho}^{\wedge\pm}(\eta) = k_\nu(\xi)\hat{f}_{j+2\nu\rho}^{\pm}(\eta), \quad 1\le j\le 2\rho,\ 1\le\nu\le s.$$

Remark 6. (i) The condition (L.2) can be removable. (ii) We can prove the principles of limiting amplitude and limiting absorption for the operator $\mathbb{A}$ by Theorem 5 and representations of eigenfunctions (see [9]).

4. Outline of proof

The self-adjoint operator $\mathbb{A}$ admits a uniquely determined spectral resolution:

$$(23) \qquad \mathbb{A} = \int_{-\infty}^{\infty} \lambda dE(\lambda),$$

where $\{E(\lambda)\}_{-\infty<\lambda<\infty}$ denotes the right-continuous spectral family of $\mathbb{A}$. Then it follows from the Stieltjes inversion formula that for $f\in C_0^\infty(R_+^n)$ and $a<b$

$$(24) \qquad (\{(E(b)+E(b-0))/2 - (E(a)+E(a-0))/2\}f,f)$$

$$= \lim_{\varepsilon\downarrow 0} \pi^{-1}\Sigma_{j=1}^{2\rho} \int_{\Xi^n} d\eta \int_a^b dk \frac{\varepsilon}{(\lambda_j(\eta)-k)^2+\varepsilon^2}|\hat{f}_j(\eta;k\pm i\varepsilon)|^2,$$

where (,) denotes the inner product of $L^2(R_+^n)$ and

$$(25) \qquad \hat{f}_j(\eta;\lambda) = \int_{R_+^n} \Psi_j(x,\eta;\lambda)^* f(x)dx, \quad \mathrm{Im}\ \lambda\neq 0,\ 1\leq j\leq 2\rho.$$

In order to prove the expansion theorem it suffices to show that we can interchange the order of $\lim_{\varepsilon\downarrow 0}$ and $\int_{\Xi^n} d\eta$ in (24). On the other hand we have

$$(26) \qquad \Psi_j(x,\eta;\lambda) = (2\pi)^{-n/2}e^{ix\cdot\eta}P_j(\eta)$$

$$- \frac{1}{i}(2\pi)^{-1/2}\overline{\mathcal{F}}_{y'}[G(x,y',+0;\lambda)](\xi)A_nP_j(\eta).$$

Thus, the part most involved of our study is to analyse the behavior around the singular points of the second term on the right hand side of (26) where the Lopatinski determinant vanishes. The detailed proof and further results are given in [8].

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